Analytic continuation of Dirichlet series with almost periodic coefficients

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November 9, 2008

Abstract

We consider Dirichlet series $\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} g(n\alpha)e^{-\lambda_n s}$ for fixed irrational α and periodic functions g. We demonstrate that for Diophantine α and smooth g, the line $\mathrm{Re}(s) = 0$ is a natural boundary in the Taylor series case $\lambda_n = n$, so that the unit circle is the maximal domain of holomorphy for the almost periodic Taylor series $\sum_{n=1}^{\infty} g(n\alpha)z^n$. We prove that a Dirichlet series $\zeta(s) = \sum_{n=1}^{\infty} g(n\alpha)/n^s$ has an abscissa of convergence $\sigma_0 = 0$ if g is odd and real analytic and α is Diophantine. We show that if g is odd and has bounded variation and α is of bounded Diophantine type r, the abscissa of convergence is smaller or equal than $1 - 1/r$. Using a polylogarithm expansion, we prove that if g is odd and real analytic and α is Diophantine, then the Dirichlet series $\zeta(s)$ has an analytic continuation to the entire complex plane.

AMS classification: 11M99, 30D99, 33E20

1 Introduction

Let $g : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous 1-periodic L^2 function with Fourier expansion $g(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$. Define the ζ function

$$\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s} .$$

For irrational α, we call this a **Dirichlet series with almost periodic coefficients**. An example is the Clausen function, where $g(x) = \sin(2\pi x)$ or the poly-logarithm, where $g(x) = \exp(2\pi i x)$. Another example arises with $g(x) = x - \lfloor x + 1/2 \rfloor$, the signed distance from x to the nearest integer. Obviously for $\mathrm{Re}(s) > 1$, such a Dirichlet series converges uniformly to an analytic limit.

The case that α is rational is less interesting, as the following computation illustrates.

For periodic $\alpha = p/q$ and any odd function g, the zeta function has an abscissa of convergence 0 and allows an analytic continuation to the entire plane.

Proof. Write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{g(np/q)}{n^s} &= \sum_{\ell=1}^{q} \sum_{n=0}^{\infty} \frac{g((nq+\ell)(p/q))}{(nq+\ell)^s} \\ &= \frac{1}{q^s} \sum_{\ell=1}^{q} \sum_{n=0}^{\infty} \frac{g(n+\ell p/q)}{(n+\ell/q)^s} = \frac{1}{q^s} \sum_{\ell=1}^{q} g(\ell/q)\zeta(s,\ell/q) ,\end{aligned}$$

where $\zeta(s,u) = \sum_{n=1}^{\infty} 1/(n+u)^s$ is the Hurwitz zeta function. So, the periodic zeta function is a just finite sum of Hurwitz zeta functions which individually allow a meromorphic continuation. Each Hurwitz zeta function is analytic everywhere except at 1, where it has a pole of residue 1: the series $\zeta(s,u) - 1/(s-1)$ has the abscissa of convergence 0 and allows an analytic extension to the plane. So, if $\sum_{n=1}^{q} g(n\alpha) = 0$, which is the case for example if g is odd, then the periodic Dirichlet series has abscissa of convergence 0 and admits an analytic continuation to the plane. $\square$

When $\alpha = 0$, the function $\zeta_{g,\alpha}$ is merely a multiple of the Riemann zeta function.

An other special case $\lambda_n = n$ leads to Taylor series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $z = e^{-s}$. Also here, the rational case is well understood:

If $\alpha = p/q$ is rational, the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ has a meromorphic extension to the entire plane.

Proof. If $\alpha = p/q$ define $h(z) = \sum_{n=1}^{q} g(n\alpha) z^n$. Then

$$f(z) = h(z)(1 + z^q + z^{2q} + \dots) = \frac{h(z)}{1-z^q} .$$

The right hand side provides the meromorphic continuation of f. $\square$

We usually assume $\int g \, dm = 0$ because we are interested in the growth of the random walk in the case $s = 0$ and because if $\int g \, dm \neq 0$, the abscissa of convergence is in general 1: for $f(x) = a_0 + \sin(x)$ for example, where the Dirichlet series with $\lambda_n = \log(n)$ is a sum of the standard zeta function and the Clausen function. The abscissa of convergence is $\sigma_0 = 1$ except for $a_0 = 0$, where the abscissa drops to 0.

Zeta functions $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ can more generally be considered for any dynamically generated sequence $a_n = g(T^n x)$, where T is a homeomorphism of a compact topological space X and g is a continuous function and λ_n grows monotonically to ∞.

Random Taylor series associated with an ergodic transformation were considered in [3, 6]. The topic has also been explored in a probabilistic setup, where a_n are independent random symmetric variables, in which case the line $\mathrm{Re}(s) = \sigma_0$ is a natural boundary [11]. Analytic continuation questions have also been studied

for other functions: if the coefficients are generated by finite automata, a meromorphic continuation is possible [8].

In this paper we focus on Taylor series and ordinary Dirichlet series. We restrict ourselves to the case, where the dynamical system is an irrational rotation $x \mapsto x + \alpha$ on the circle. The minimality and strict ergodicity of the system will often make the question independent of the starting point $x \in X$ and allow to use techniques of Fourier analysis and the Denjoy-Koksma inequality. We are able to make statements if α is Diophantine.

Dirichlet series can allow to get information on the growth of the random walk $S_k = \sum_{n=1}^k g(T^n(x))$ for a m-measure preserving dynamical system $T : X \to X$ if $\int_X g(x)\, dm(x) = 0$. Birkhoffs ergodic theorem assures $S_k = o(k)$. Similarly as the law of iterated logarithm refines the law of large numbers in probability theory, and Denjoy-Koksma type results provide further estimates on the growth rate in the case of irrational rotations, one can study the growth rate for more general dynamical systems. The relation with algebra is as follows: if S_k grows like k^β then the abscissa of convergence of the ordinary Dirichlet series is smaller or equal to β. In other words, establishing bounds for the analyticity domain allows via Bohr's theorem to get results on the abscissa of convergence which give upper bounds of the growth rate. Adapting the λ_n to the situation allows to explore different growth behavior. The algebraic concept of Dirichlet series helps so to understand a dynamical concept.

2 Almost periodic Taylor series

A general Dirichlet series is of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} .$$

For $\lambda_n = \log(n)$ this is an ordinary Dirichlet series, while for $\lambda_n = n$, it is a Taylor series $\sum_n a_n z^n$ with $z = e^{-s}$. We primarily restrict our attention to these two cases.

We begin by considering the easier problem of Taylor series with almost periodic coefficients and examine the analytic continuation of such functions beyond the unit circle. Given a non-constant periodic function g, we can look at the problem of whether the Taylor series

$$f(z) = \sum_{n=1}^{\infty} g(n\alpha) z^n$$

can be analytically continued beyond the unit circle. Note that all these functions have radius of convergence 1 because $\limsup_n |g(n\alpha)|^{1/n} = 1$.

We have already seen in the introduction that if $\alpha = p/q$ is rational, the function f has a meromorphic extension to the entire plane. There is an other case where analytic continuation can be established immediately:

Lemma 1. *If g is a trigonometric polynomial and α is arbitrary, then f has a meromorphic extension to the entire plane.*

Proof. Since $g(x) = \sum_{n=1}^{k} c_n e^{2\pi i n x}$, it is enough to verify this for $g(x) = e^{2\pi i n x}$, in which case the series sums to $f(z) = 1/(1 - e^{2\pi i n \alpha} z)$. $\square$

On the other hand, if infinitely many of the Fourier coefficients for g are nonzero, analytic continuation may not be possible.

Proposition 2. *Fix $r > 1$. Assume g is in C^t for $t > 2r+1$, and that all Fourier coefficients c_k of g are nonzero and that α is of Diophantine type r. Then the almost periodic Taylor series $f_{g,\alpha}(z) = \sum_{n=0}^{\infty} g(n\alpha) z^n$ can not be continued beyond the unit circle.*

Proof. Write

$$
\begin{aligned}
f_{g,\alpha}(z) &= \sum_{n=0}^{\infty} g(n\alpha) z^n = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k n \alpha} z^n \\
&= \sum_{k=-\infty}^{\infty} c_k \sum_{n=0}^{\infty} e^{2\pi i k n \alpha} z^n = \sum_{k=-\infty}^{\infty} \frac{c_k}{1 - e^{2\pi i k \alpha} z} .
\end{aligned}
$$

Fix some j and consider the radial limit

$$
\begin{aligned}
\lim_{t \to 1^-} f_{g,\alpha}(t e^{2\pi i j \alpha}) &= \lim_{t \to 1^-} \sum_{k=-\infty}^{\infty} \frac{c_k}{1 - t e^{2\pi i k \alpha} e^{2\pi i j \alpha}} \\
&= \lim_{t \to 1^-} \left(\frac{c_j}{1-t} + \sum_{\substack{k=-\infty \\ k \neq -j}}^{\infty} \frac{c_k}{1 - t e^{2\pi i (k+j) t}} \right) .
\end{aligned}
$$

The latter sum converges at $t = 1$, as by the Diophantine condition, the denominator is bounded below by $(n-k)^{2r}$, while the numerator c_k is bounded above by $1/k^t$ for $t > 2r+1$ by the differentiability assumption on g. Because the term $c_j/(1-t)$ diverges for $t \to 1^-$, it follows that this radial limit is infinite for all j. Consequently, f does not admit an analytic continuation to any larger set. $\square$

Remark 1. This result is related to a construction of Goursat, which shows that for any domain D in $\mathbb{C}$, there exists a function which has D as a maximal domain of analyticity [14]. In contrary to lacunary Taylor series like $\sum_{j=1}^{\infty} z^{2^j}$ which have the unit circle as a natural boundary too, all Taylor coefficients are in general nonzero in the Taylor series of Proposition 2.

Remark 2. The function

$$
f(z) = \sum_{k=-\infty}^{\infty} \frac{c_k}{1 - e^{2\pi i k \alpha} z} = \sum_{k=-\infty}^{\infty} \frac{a_k}{z - z_k}
$$

is defined also outside the unit circle. The subharmonic function $\log(f)(z) = \int \log|z-w| \, dk(w)$ has a Riesz measure dk supported on the unit circle which is dense pure point.

Remark 3. The requirement $g \in C^t$ is by no means necessary. For example, any nonconstant step function g is not even continuous, but by Szegő's theorem (see [14]) for power series with finitely many distinct coefficients which do not eventually repeat periodically, $f(z) = \sum_{n=1}^{\infty} g(n\alpha) z^n$ can not be analytically extended beyond the unit disk.

Remark 4. The condition that all Fourier coefficients are nonzero may be relaxed to the assumption that the set of $e^{2\pi i k\alpha}$, where k ranges over the indices of nonzero Fourier coefficients, is dense in S^1.

As the last remark may suggest, trigonometric polynomials are not the only functions whose associated series allow an analytic continuation beyond the unit circle.

Proposition 3. *Let $K \subset \{|z| = 1\}$ be an arbitrary closed set on the unit circle. There exists an almost periodic Taylor series which has an analytic continuation to $\mathbb{C} \setminus K$ but not to any point of K.*

Proof. Set

$$c_k = \begin{cases} 0 & \text{if } e^{2\pi i\alpha k} \in K, \\ 1/k! & \text{otherwise .} \end{cases}$$

and let $g(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$, and let α be Diophantine of any type $r > 1$. Inside the unit circle we have

$$f_{g,\alpha}(z) = \sum_{k=-\infty}^{\infty} \frac{c_k}{1 - e^{2\pi i k\alpha} z}.$$

For any j for which $e^{2\pi i j\alpha} \in K$, this sum converges uniformly on a closed ball around z which does not intersect K (since the denominators of the non-vanishing terms are uniformly bounded), and thus has an analytic neighborhood around such $e^{2\pi i j\alpha}$. Any point in K lies in a compact neighborhood of such a point. On the other hand, by the arguments of the preceding lemma, analytic continuation is not possible in K itself. □

3 Ordinary Dirichlet series

Cahen's formula for the abscissa of convergence of an ordinary Dirichlet series $\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} a_n/n^s$ is

$$\sigma_0 = \limsup_k \frac{\log S_k}{\log k},$$

if $S_k = \sum_{n=1}^{k} g(n\alpha)$ does not converge [4]. We will compute the abscissa of convergence for two classes of functions and so derive bounds on the random walks S_k which are stronger than those implied by the Denjoy-Koksma inequality.

The first situation applies to real analytic g, where we can invoke the cohomology theory of cocycles over irrational rotations:

Proposition 4. *If g is real analytic, $\int g(x)\,dx = 0$ and α is Diophantine, then the series for $\zeta_{g,\alpha}(s)$ converges and is analytic for* $\mathrm{Re}(s) > 0$. *In other words, the abscissa of convergence is* 0.

Proof. Since $g_0 = 0$, $g = \sum_{n=1}^{\infty} g_n e^{2\pi i n x}$ is an additive coboundary: the Diophantine property of α implies that the real analytic function

$$h(x) = \sum_{n=1}^{\infty} g_n \frac{e^{2\pi i n x}}{(e^{2\pi i n\alpha} - 1)}$$

solves

$$g(x) = h(x+\alpha) - h(x) \mod 1 \ .$$

Because $S_k = \sum_{n=1}^{k} g(n\alpha)$ does not converge, but stays bounded in absolute value by $2||h||_\infty$, Cahen's formula immediately implies that $\sigma_0 = 0$.
Lets give a second proof without Cahen's formula. For $0 < \mathrm{Re}(s) < 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s} &= \sum_{n=1}^{\infty} \frac{h((n+1)\alpha)}{n^s} - \frac{h(n\alpha)}{n^s} \\
&= -h(\alpha) + \sum_{n=2}^{\infty} h(n\alpha)[1/(n-1)^s - 1/n^s] \\
&= -h(\alpha) + \sum_{k=2}^{\infty} h(n\alpha)\frac{n^s - (n-1)^s}{n^s(n-1)^s} \\
&\leq ||h||(1 + \sum_{k} \frac{n^s - (n-1)^s}{n^s(n-1)^s}) \ .
\end{aligned}$$

The sum is bounded for $1 > \mathrm{Re}(s) > 0$ because $|(n+1)^s - n^s| \leq |sn^{s-1}|$ so that $((n+1)^s - n^s)/((n+1)^s n^s) \leq |s|\, n^{-1}(n+1)^{-s} < |s|\, n^{-1-s}$. The function $\zeta_{g,\alpha}(s)$ is analytic in $\mathrm{Re}(s) > 0$ as the limit of a sequence of analytic functions which converge uniformly on a compact subset of the right half plane. The uniform convergence follows from Bohr's theorem (see [4], Theorem 52). $\square$

We do not know what happens for irrational α which are not Diophantine except if g is a trigonometric polynomial:

Lemma 5. *If g is a trigonometric polynomial of period 1 with $\int_0^1 g(x)\,dx = 0$ and α is an arbitrary irrational number, then the abscissa of convergence of $\zeta_{g,\alpha}$ is* 0.

Proof. If g is a trigonometric polynomial, then g is a coboundary for **every** irrational α because $e^{2\pi i n x} = h(x+\alpha) - h(x)$ for $h(x) = e^{2\pi i n x}/(e^{2\pi i n x} - 1)$. In the case of the Clausen function for example and $s = 0$, we have $\sum_{k=0}^{n-1} \sin(2\pi k\alpha) = \mathrm{im}(e^{2\pi i n\alpha} - 1)/(e^{2\pi i\alpha} - 1)$. It follows that the series $\zeta(s)$ converges for all trigonometric polynomials g, for **all** irrational $\alpha \neq 0$ and all $\mathrm{Re}(s) > 0$. $\square$

Remark 5. A special case is the zeta function

$$\sum_{n=1}^{\infty} e^{2\pi i n\alpha}/n^s$$

which is can be written as $\psi_s(e^{2\pi i\alpha})$, where ψ_s is the **polylogarithm**. Integral representations like

$$\psi_s(z) = \frac{z}{\Gamma(s)} \int_0^1 [\log(1/t)]^{s-1} \frac{dt}{1-zt}$$

(see i.e. [12]) show the analytic continuation for $z \neq 1$ rsp. $\alpha \neq 0$. It follows that the function $\zeta_{g,\alpha}$ has an analytic continuation to the entire complex plane for all irrational α if g is a trigonometric polynomial. We will use polylogarithms later.

4 The bounded variation case

The result in the last section had been valid if α satisfies some Diophantine condition and g is real analytic. If the function g is only required to be of bounded variation, then the abscissa of convergence can be estimated. Lets first recall some definitions:

A real number α for which there exist $C > 0$ and $r > 1$ satisfying

$$|\alpha - \frac{p}{q}| \geq \frac{C}{q^{1+r}}$$

for all rational p/q is called **Diophantine of type** r. The set of real numbers of type r have full Lebesgue measure for all $r > 1$ so that also the intersection of all these types have. The **variation** of a function f is $\sup_P \sum_i |f(x_{i+1}) - f(x_i)|$, where the supremum is taken over all partitions $P = \{x_1, \dots, x_n\}$ of $[0,1]$.

Proposition 6. *If α is Diophantine of type $r > 1$ and if g is of bounded variation with $\int_0^1 g(x)\ dx = 0$, then the series for $\zeta_{g,\alpha}(s)$ has an abscissa of convergence $\sigma_0 \leq (1 - 1/r)$.*

Proof. The Denjoy-Koksma inequality (see Lemma 12) implies that for any m, n, the sum $S_{n,m} = \sum_{k=n}^m g(k\alpha)$ satisfies the estimate $S_{n,m} \leq C \log(m-n)\ (m-n)^{1-1/r}$, with $C = \mathrm{Var}(g)$. Cahen's formula for the abscisse of convergence gives

$$\limsup_{n\to\infty} \frac{\log S_{1,n}}{\log(n)} \leq 1 - \frac{1}{r}\ .$$

□

Remark 6. A weaker result could be obtained directly, without Cahen's formula. Choose $\ell_k \to \infty$ such that

$$U_k = \sum_{j=\ell_k}^{l_{k+1}} [g(j\alpha)/\ell_k^s] \leq C \log(\ell_{k+1} - \ell^k)(\ell_{k+1} - \ell^k)^{1-1/r}/\ell_k^s$$

$$V_k = \sum_{j=\ell_k}^{l_{k+1}} [\frac{g(j)}{\ell_k^s} - \frac{g(j)}{j^s}] \leq \frac{(\ell_{k+1} - \ell_k)^2}{l_{k+1}^s \ell_k^s} ||g||$$

are summable. Then

$$S_k = \sum_{j=\ell_k}^{l_{k+1}} g(j\alpha)/j^s \leq U_k + V_k$$

is summable.

To compare: if $g(T^n x)$ are independent, identically distributed random variables with mean 0 and finite variance a, then $\sum_{k=1}^n g(T^k x)$ grows by the law of iterated logarithm like $a\sqrt{n}\log\log(n)$ and the zeta function converges with probability 1 for $\mathrm{Re}(s) > 1/2$. The following reformulation of the law of iterated logarithm follows directly from Cahen's formula:

(Law of iterated logarithm) If T is a Bernoulli shift such that $g(x) = x_0$ produces independent identically distributed random variables $g(T^n x)$ with finite nonzero variance, then the Dirichlet series $\zeta_{g,T}$ has the abscissa of convergence $1/2$.

We do not have examples in the almost periodic case yet, where the abscissa of convergence is strictly between 0 and 1 like $\sigma_0 = 1/2$.

5 Analytic continuation

The **Lerch transcendent** is defined as

$$L(z,s) = \sum_{n=0}^{\infty} z^n/(n+a)^s \ .$$

For $z = e^{2\pi i\alpha}$, we get the **Lerch zeta function**

$$L(\alpha,s) = \sum_{n=0}^{\infty} e^{2\pi i n\alpha}/(n+a)^s \ .$$

In the special case $a = 1$ we have the **polylogarithm** $L(z,s) = \sum_{n=1}^{\infty} z^n/n^s$. For $z = 1$ and general a, we have the **Hurwitz zeta function**, which becomes for $a = 1, z = 1$ the **Riemann zeta function**. The following two Lemmas are standard (see [7]).

Lemma 7. *For and $|z| = 1$, there is an integral representation*

$$L(z,s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1 - ze^{-t}} \, dt \ .$$

For fixed $|z| = 1, z \neq 1$, this is analytic in s for $\mathrm{Re}(s) > 1$. For fixed $\mathrm{Re}(s) > 1$, it is analytic in z for $z \neq 1$.

Proof. By expanding $1/(1 - ze^{-t}) = \sum_{n=0}^{\infty} z^n/e^{nt}$, the claim is equivalent to

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}e^{-at-nt} \, dt = \frac{1}{(n+a)^s} \ .$$

A substitution $u = (a+n)t, du = (a+n)dt$ changes this to

$$\frac{1}{\Gamma(s)} \int_0^{\infty} u^{s-1}e^{-u} \frac{1}{(a+n)^s} \, du = \frac{1}{(n+a)^s} \ .$$

Now use $\Gamma(s) = \int_0^{\infty} u^{s-1}e^{-u} \, du$.

The improper integral is analytic in s because $|1-ze^{-t}|$ is $\geq \sin(\arg(z))$ for $\mathrm{Re}(z) \geq 0$ and ≥ 1 for $\mathrm{Re}(z) \leq 0$ and for $\sigma = \mathrm{Re}(s) > 1$, we have

$$|L(z,s)| \leq \frac{1}{|\Gamma(s)|} \int_0^\infty |t^{\sigma-1} e^{-at}| \, dt \frac{1}{|1-z|} .$$

□

Lemma 8. *The Lerch transcendent has for fixed $|z|=1, z \neq 1$ and $a > 0$ an analytic continuation to the entire s-plane. In every bounded region G in the complex plane, there is a constant $C = C(G,a)$ such that $|L(z,s)| \leq C/|z-1|$ and $|(\partial_z)^n L(z,s)| \leq Cn!/|z-1|^{n+1}$.*

Proof. For any bounded region G in the complex plane we can find a constant C such that

$$|\int_0^\infty \frac{t^{s-1}e^{-at}}{1-ze^{-t}} \, dt| \quad \leq \quad (\int_0^\infty |t^{s-1}e^{-at}| \, dt) \max_t \frac{1}{|1-ze^{-t}|} \leq \frac{C}{|1-z|} .$$

Similarly, we can estimate $|\partial_z^n L(z,s)| \leq Cn!/|z-1|^n$ for any integer $n > 0$. The identity

$$L(z,s-1) = (a + z\partial_z) L(z,s) \tag{1}$$

allows us to define L for $\mathrm{Re}(s) < 1$: first define L in $0 < \mathrm{Re}(s) < 1$ by the recursion (1). Then use the identity (1) again to define it in the strip $-1 < \mathrm{Re}(s) < 0$, then in the strip $-2 < \mathrm{Re}(s) < 1$, etc. □

Remark 7. The Lerch transcendent is often written as a function of three variables:

$$\phi(x,a,s) = \sum_{n=0}^\infty e^{2\pi i x}/(n+a)^s .$$

it satisfies the functional equation

$$\begin{aligned} (2\pi)^s \phi(x,a,1-s) &= \Gamma(s) \exp(2\pi i(s/4 - ax)) \phi(x,-a,s) \\ &+ \Gamma(s) \exp(2\pi i(-s/4 + a(1-x))) \phi(1-x,a,s)) . \end{aligned}$$

See [13]. Using the functional equation to do the analytic continuation is less obvious.

One of the main results in this paper is the following theorem:

Theorem 9. *For all Diophantine α and every real analytic periodic function g satisfying $\int g(x) \, dx = 0$, the series $\zeta_{g,\alpha}(s) = \sum_{n=1}^\infty g(n\alpha)/n^s$ has an analytic continuation to the entire complex plane.*

Proof. The Fourier expansion of g evaluated at $x = n\alpha$ gives

$$g(n\alpha) = \sum_{k=-\infty}^\infty c_k e^{2\pi i n k \alpha} .$$

It produces a polylog expansion of $f = \zeta_{g,\alpha}$

$$f(s) = \sum_{k=-\infty}^\infty c_k L(e^{2\pi i k \alpha}, s)$$

with $a = 1$. Because g is real analytic, there exists $\delta > 0$ such that $|c_k| \leq e^{-|k|\delta}$. Since $k \neq 0$ and α is Diophantine, $|e^{2\pi i k\alpha} - 1| \geq c/|k|^r$ so that $L(e^{2\pi i k\alpha}, s) \leq |k|^r/c$ and

$$|f(s)| = \sum_{k=-\infty}^{\infty} |c_k||L(e^{2\pi i k\alpha}, s)| \leq \sum_{k=-\infty}^{\infty} e^{-|k|\delta} k^r/c < \infty .$$

□

6 A commutation formula

For any periodic function g, the series

$$T(g) = \sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s}$$

produces for fixed s in the region of convergence a new periodic function in α. For fixed α it is a Dirichlet series in s. The Clausen function is $T(\sin 2\pi x)$ and the polylogarithm is $T(\exp(2\pi i x))$. We may then consider a new almost periodic Dirichlet series generated by this function, defined by $T(T(g)(s))(t)) = T(T(g))(s,t)$.

The following commutation formula can be useful to extend the domain, where Dirichlet series are defined:

Lemma 10 (Commutation formula)**.** *For* $\mathrm{Re}(s) > 1$ *and* $\mathrm{Re}(t) > 1$*, we have*

$$T(T(g))(s,t) = T(T(g))(t,s)) .$$

Proof. We have

$$T(g)(s) = \sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s} ,$$

which we regard as a periodic function in α. It is continuous in α if evaluated for fixed $\mathrm{Re}(s) > 1$. Then where our sum converges absolutely (which holds at least for $s, t > 1$),

$$\begin{aligned} T(T(g))(s,t) &= \sum_{m=1}^{\infty} \frac{T(g)(s)(m\alpha)}{n^t} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^s m^t} g(mn\alpha) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^s m^t} g(mn\alpha) = T(T(g))(t,s). \end{aligned}$$

□

In fact, the double sum can be expressed as a single sum using the **divisor**

sum function $\sigma_t(k) = \sum_{m|k} m^t$:

$$T(T(g))(s,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^s m^t} g(mn\alpha) = \sum_{k=1}^{\infty} \left(\left(\sum_{m|k} \frac{m^{t-s}}{k^t} \right) g(k\alpha) \right)$$
$$= \sum_{k=1}^{\infty} \frac{\sigma_{t-s}(k)}{k^t} g(k\alpha) \; . \tag{2}$$

We can formulate this as follows: for $s, t > 1$, we have

$$\zeta_{g_s,\alpha}(t) = \zeta_{g_t,\alpha}(s) \; .$$

For example, evaluating the almost periodic Dirichlet series for the periodic function $g_3(x) = \sum_{k=1}^{\infty}(1/k^3)\sin(2\pi kx)$ at $s = 5$ is the same as evaluating the almost periodic Dirichlet series of the periodic function $g_5(x) = \sum_{k=1}^{\infty}(1/k^5)\sin(kx)$ and evaluating it at $s = 3$. But since $g_3(x)$ is of bounded variation, the Dirichlet series $T(g_3)$ has an analytic continuation to all $\mathrm{Re}(s) > 0$ and $T(g_3)(0.5)$ for example is defined if α is Diophantine of type $1 < r < 2$. The commutation formula allows us to define $T(g_{0.5})(3) = \zeta_{g_{0.5}}(3)$ as $\zeta_{g_3}(0.5)$, even so $g_{0.5}$ is not in $L^2(T^1)$.

Remark 8. The commutation formula generalizes. The irrational rotation $x \mapsto x + \alpha$ on $X = S^1$ can be replaced by a general topological dynamical system on X.

In the particular case that f is the Clausen function $g = T(\sin)$, the expression (2) is itself a Fourier series, whose coefficients are the divisor function σ. We have $g(x) = \sum_{n=1}^{\infty} \frac{1}{n^t}\sin(2\pi nx)$ and $t > s$: the value of $\zeta(s)$ as a function of α is the Fourier transform on $l^2(\mathbb{Z})$ of the multiplicative arithmetic function $h(k) = \sigma_{t-s}(k)/k^s$. For $t = 2, s = 1$ for example, we get

$$\sum_n \frac{g(n\alpha)}{n} = \sum_{k=1}^{\infty} \frac{\sigma(k)}{k} \sin(2\pi k\alpha)$$

if g is the function with Fourier coefficients $1/n^2$. In that case, the Fourier coefficients of the function has the multiplicative function $\sigma(n)/n$ (called the **index** of n) as coefficients. For $t = 1$, we have $\sum_{n=1}^{\infty} \frac{1}{n}\sin(2\pi nx) = 1/2 - (x \bmod 1)$ and for odd integer $t > 1$, the function

$$g_k(x) = \sum_{k=1}^{\infty} \sin(2\pi kx)/k^t$$

is a **Bernoulli polynomial**.

For $t = s$, we get

$$\sum_n \frac{g(n\alpha)}{n^s} = \sum_{k=1}^{\infty} \frac{d(k)}{k^s} \sin(2\pi k\alpha) \; ,$$

where $d(k)$ is the number of divisors of k. These sums converge absolutely for $\mathrm{Re}(s) > 1$. If t is a positive odd integer, g is a Bernoulli polynomial (e.g. for $t = 3$,

we have $g(x) = (\pi^3/3)(x - 3x^2 + 2x^3))$. For $s = 2$ (and still $t = 3$), we have

$$\sum_n \frac{g(n\alpha)}{n^2} = \sum_{k=1}^{\infty} \frac{\sigma(k)}{k^2} \sin(2\pi k\alpha) \ .$$

The functions $f_{s,t}(\alpha) = T(T(\sin))(s,t)$, regarded as periodic functions of α, may be related by an identity of Ramanujan [15]. Applying Parseval's theorem to these Fourier series, one can deduce

$$\begin{aligned} \int_0^1 f_{s,t}(\alpha) f_{u,v}(\alpha)\, d\alpha &= \sum_{n=1}^{\infty} \frac{\sigma_{t-s}(n)}{n^t} \frac{\sigma_{v-u}(n)}{n^v} \\ &= \frac{\zeta(s+u)\zeta(s+v)\zeta(t+u)\zeta(t+v)}{\zeta(s+t+u+v)} \ . \end{aligned}$$

7 Unbounded variation

If g fails to be of bounded variation, the previous results do not apply. Still, there can be boundedness for the Dirichlet series if $\mathrm{Re}(s) > 0$. The example $g(x) = \log|2 - 2\cos(2\pi x)| = 2\log|e^{2\pi ix} - 1|$ appears in the context of KAM theory and was the starting point of our investigations. The product $\prod_{k=1}^n |2\cos(2\pi k\alpha) - 2|$ is the determinant of a truncated diagonal matrix representing the Fourier transform of the Laplacian $L(f) = f(x+\alpha) - 2f(x) + f(x-\alpha)$ on $L^2(\mathbb{T})$.

The function $g(x)$ has mean 0 but it is not bounded and therefore has unbounded variation. Numerical experiments indicate however that at least for many α of constant type, $|\sum_{k=0}^{n-1} g(k\alpha)| \leq C\log(n)$. We can only show:

Proposition 11. *If α is Diophantine of type $r > 1$, then for $g(x) = \log|2 - 2\cos(2\pi x)|$,*

$$\sum_{n=0}^{k-1} g(n\alpha) \leq C k^{1-1/r} (\log(k))^2$$

and $\zeta(s)$ converges for $\mathrm{Re}(s) > 0$.

Proof. $\prod_{j=1}^{q-1} |e^{2\pi ij/q} - 1| = q$ because

$$\prod_{j=1}^{q-1} |e^{2\pi ij/q} - z| = \frac{z^q - 1}{z - 1} = \sum_{j=0}^{q-1} z^j$$

which gives $\prod_{j=1}^{q-1} |e^{2\pi ij/q} - 1| = q$. Define $\log^M(x) = \max(-M, \log(x))$. Now

$$\sum_{j=1}^{q-1} \log^M(|e^{2\pi ij\alpha} - 1|) \leq M\log(q)$$

for all q and also for general k by the classical Denjoy-Koksma inequality. Choose $M = 3\log(q)$. Then the set $Y_M = \{\log(|\sin(2\pi x)|) < -M\ \} = \{|\sin(2\pi x)| < e^{-M}\ \} \subset \{|x| < 2e^{-M} = 2q^{-3}\}$. The finite orbit $\{k\alpha\}_{k=1}^{q-1}$ never hits that set

and the sum is the same when replacing $\log^M$ with the untruncated log. We get therefore

$$\sum_{j=1}^{q-1} \log(|e^{2\pi ij\alpha}-1|) = \sum_{j=1}^{q-1} \log^M(|e^{2\pi ij\alpha}-1|) \leq 3\log(q)^2 \ .$$

The rest of the proof is the same as for the classical Denjoy-Koksma inequality. $\square$

The Denjoy-Koksma inequality is treated in [5, 1]. Here is the exposition as found in [9].

Lemma 12 (Jitomirskaja's formulation of Denjoy-Koksma)**.** *Assume α is Diophantine of type $r>1$ and g is of bounded variation and $\int_0^1 g(x)\ dx = 0$. Then $S_k = \sum_{n=1}^k g(n\alpha)$ satisfies*

$$|S_k| \leq C k^{1-1/r} \log(k) \mathrm{Var}(g) \ .$$

If α is of constant type and g is of bounded variation and $\int_0^1 g(x)\ dx = 0$, then $S_k \leq C\log(k)$.

Proof. (See [9], Lemma 12). If p/q is a periodic approximation of α, then

$$|S_q| \leq \mathrm{Var}(f) \ .$$

To see this, divide the circle into q intervals centered at the points $y_m = mp/q$. These intervals have length $1/q \pm O(1/q^2)$ and each interval contains exactly one point of the finite orbit $\{y_k = k\alpha\}_{k=1}^q$. Renumber the points so that y_m is in I_m. By the intermediate value theorem, there exists a Riemann sum $\frac{1}{q}\sum_{i=0}^{q-1} f(x_i) = \int f(x)\ dx = 0$ for which every x_i is in an interval I_i (choosing the point $x_i = \min_{x\in I_i} f(x)$ gives an lower and $x_m = \max_{x\in I_m} f(x)$ gives an upper bound). If $\sum_{j=0}^{q-1} f(y_j) - f(x_j) \leq \sum_{j=0}^{q-1} |f(y_j)-f(x_j)| + |f(x_j)-f(y_{j+1})| \leq \mathrm{Var}(f)$.
Now, if $q_m \leq k \leq q_{m+1}$ and $k = b_m q_m + b_{m-1}q_{m-1} + \cdots + b_1 q_1 + b_0$, then

$$S_k \leq (b_0 + \cdots + b_m)\mathrm{Var}(f) \leq \sum_{i=0}^m \frac{q_{i+1}}{q_i}\mathrm{Var}(f)$$

because $b_j \leq q_{j+1}/q_j$.
If α is of constant type then $\frac{q_{i+1}}{q_i}$ is bounded and $m < 2\log(k)/\log(2)$ implies $S_k \leq (2\log(k)/\log(2))\mathrm{Var}(f)$.
If α is Diophantine of type $r>1$, then $||q\alpha|| \leq c/q^r$ and $q_{i+1} \leq q_i^r/c$ which implies $q_{i+1}/q_i < q_{i+1}^{1-1/r}/c^{1/r}$ and so

$$|S_k| \leq (c^{-1/r}\sum_{i=1}^m q_i^{1-1/r} + \frac{k}{q_m})\mathrm{Var}(f) \leq (c^{-1/r} m q_m^{1-1/r} + \frac{k}{q_m})\mathrm{Var}(f) \ .$$

The general fact $m \leq 2\log(q_m)/\log(2) \leq 2\log(k)/\log(2)$ deals with the first term. The second term is estimated as follows: from $k \leq q_{m+1} \leq q_m^r/c$, we have $q_m \geq (ck)^{1/r}$ and $k/q_m \leq c^{-1/r}k^{1-1/r}$. $\square$

Remark 9. One knows also $|S_n| \leq C\log(n)^{2+\epsilon}$ if the continued fraction expansion $[a_0, a_1, ...]$ of α satisfies $a_m < m^{1+\epsilon}$ eventually. See [2].

8 Questions

We were able to get entire functions $\zeta(s) = \sum_n g(n\alpha)/n^s$ for rational α and Diophantine α. What happens for Liouville α if g is not a trigonometric polynomial? What happens for more general g?

For every g and s we get a function $\alpha \to \zeta_{g,\alpha}(s)$. For

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n x)$$

and Diophantine α, where

$$h(\alpha) = \zeta_{g,\alpha}(1) = \sum_{n=1}^{\infty} \frac{d(n)}{n} \sin(2\pi n\alpha) ,$$

we observe a self-similar nature of the graph. Is the Hausdorff dimension of the graph of h not an integer?

One can look at the problem for more general dynamical systems. Here is an example: for periodic Dirichlet series generated by an ergodic translation on a two-dimensional torus with a vector (α, β), where $\alpha, \alpha/\beta$ are irrational, the series is

$$\sum_{n=1}^{\infty} \frac{g(n\alpha, n\beta)}{n^s} .$$

In the case $s = 0$, this leads to the Denjoy-Koksma type problem to estimate the growth rate of the random walk

$$\sum_{n=1}^{\infty} g(n\alpha, n\beta)$$

which is more difficult due to the lack of a natural continued fraction expansion in two dimensions. In a concrete example like $g(x, y) = \sin(2\pi xy)$, the question is, how fast the sum

$$S_k = \sum_{n=1}^{k} \sin(2\pi n^2 \gamma)$$

grows with irrational $\gamma = \alpha\beta$. Numerical experiments indicate subpolynomial growth that $S_k = O(\log(k)^2)$ would hold for Diophantine γ and suggest the abscissa of convergence of the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 \gamma)}{n^s}$$

is $\sigma_0 = 0$. This series is of some historical interest since Riemann knew in 1861 (at least according to Weierstrass [10]) that $h(\gamma) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2\gamma)}{n^2}$ is nowhere differentiable.

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